Nonunique solution of the Cauchy problem for vortical flow of ideal barotropic fluid?

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Abstract

The Cuachy problem for the 3D vortical flow of ideal barotropic fluid is considered. It is shown that the solution of the Cauchy problem is unique, if one considers seven dynamic equations for seven dependent variables: the density ρ, the velocity $\mathbf{v}$ and Lagrangian variables $\boldsymbol{\xi}=\{\xi_1,\xi_2,\xi_3\}$, labeling the fluid particles. If one considers only the closed (Euler) system of four equations for four dependent variables $\rho,\mathbf{v}$, the solution is not unique. The fact is that the Euler system of hydrodynamic equations describes both the fluid motion at fixing labeling and the evolution of the fluid labeling, whose evolution is described by the Lin constraints (equations for variables $\boldsymbol{\xi}$). If one ignores the Lin constraints at the solution of the Euler system, (or one considers the Lin constraints on the basis of the solution of the Euler system), nonunique solution of the Cauchy problem is obtained.

1 Introduction

Dynamics of ideal barotropic fluid is described conventionally by the system of hydrodynamic equations

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\boldsymbol{\nabla})\,\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p, \qquad p=p(\rho)=\rho^2\frac{\partial E}{\partial \rho} \tag{1.1}$$

$$\frac{\partial \rho}{\partial t}+\boldsymbol{\nabla}(\rho\mathbf{v}) = 0 \tag{1.2}$$

where $\rho=\rho(t,\mathbf{x})$ is the fluid density, $\mathbf{v}=\mathbf{v}(t,\mathbf{x})$ is the fluid velocity, $p=p(\rho)=\rho^2E(\rho)$ is the pressure and $E(\rho)$ is the fluid internal energy per unit mass. Equations (1.1) are known as the Euler equations. The equation (1.2) is the continuity equation. Motion of the fluid particle in the given velocity field $\mathbf{v}(t,\mathbf{x})$ is described by equations

$$\frac{d\mathbf{x}}{dt}=\mathbf{v}(t,\mathbf{x}) \tag{1.3}$$

The system of seven equations (1.1) - (1.3) form the complete system of dynamic equations, describing the fluid motion. The system of equations (1.1) - (1.3) is not uniform in the sense, that equations (1.1), (1.2) are partial differential equations, whereas the equations (1.3) are ordinary differential equations.

Usually one ignores equations (1.3) and considers the system (1.1), (1.2) as dynamic equations, describing the fluid motion. Such an approach is conditioned by the fact, that the system (1.1), (1.2) is a closed system of differential equations, which can be solved without a reference to system (1.3). The solution of ordinary differential equations (1.3) is an easier problem, than a solution of partial differential equations (1.1), (1.2). If we have succeeded to solve the system (1.1), (1.2), we may hope to solve also the system (1.3). Besides, in many cases the solution of equations (1.3) is of no interest.

The idea, that taking into account solutions of equations (1.3), we may influence on the solution of the system (1.1), (1.2) seems to be delusive. However, taking into account of equations (1.3), we may transform the Euler equations (1.1) to the form which, takes into account initial conditions for variables $\mathbf{v}$ and introduce them into dynamic equations. This circumstance may be useful, if we are interested in global properties of the fluid flow, described by hydrodynamic equations (1.1), (1.2) and initial conditions together.

To carry out this idea we transform ordinary equations (1.3) to the form

$$\frac{\partial\boldsymbol{\xi}}{\partial t}+(\mathbf{v}\boldsymbol{\nabla})\boldsymbol{\xi}=0,\qquad \boldsymbol{\xi}=\{\xi_1,\xi_2,\xi_3\} \tag{1.4}$$

where $\boldsymbol{\xi}(t,\mathbf{x})=\{\xi_1(t,\mathbf{x}),\xi_2(t,\mathbf{x}),\xi_3(t,\mathbf{x})\}$ are three independent integrals of equations (1.3). The equations (1.4) are known as the Lin constraints [1]. This name appeared, when it became known, that the equations (1.4) are necessary conditons of the system (1.1), (1.2) derivation from the variational principle [2]. The system of equations (1.4) is equivalent to the system (1.3) in the sense that the system of ordinary equations

$$\frac{dt}{1}=\frac{dx^1}{v^1}=\frac{dx^2}{v^2}=\frac{dx^3}{v^3}, \tag{1.5}$$

associated with the partial differential equations (1.4), coincides with (1.3). The integrals $\boldsymbol{\xi}$ are constant on the world line of any fluid particle. If they are independent, they may label world lines of the fluid particles.

The system of seven equations (1.1), (1.2), (1.4) for seven dependent variables $\rho,\mathbf{v},\boldsymbol{\xi}$ is uniform in the sense, that all equations are the partial differential equations and all dependent variables $\rho,\mathbf{v},\boldsymbol{\xi}$ are functions of independent variables $t,\mathbf{x}$.

The system of seven equations (1.1), (1.2), (1.4) is invariant with the relabeling group of transformations

$$\xi_\alpha \rightarrow \tilde{\xi}_\alpha = \tilde{\xi}_\alpha (\boldsymbol{\xi}), \qquad \alpha = 1,2,3, \qquad \frac{\partial \left(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3\right)}{\partial (\xi_1, \xi_2, \xi_3)} \neq 0 \tag{1.6}$$

Existence of the symmetry group (1.6) for the system of seven dynamic equations (1.1), (1.2), (1.4) admits one to integrate partly the system. As a result of this integration one obtains three arbitrary functions of variables $\boldsymbol{\xi}$, and the order of the system is reduced to four equations.

Solution of the system (1.1), (1.2) with initial conditions

$$\rho (0, \mathbf{x}) = \rho_{\text{in}} (\mathbf{x}), \qquad \mathbf{v} (0, \mathbf{x}) = \mathbf{v}_{\text{in}} (\mathbf{x}) \tag{1.7}$$

may be presented in the form

$$\mathbf{v} (t, \mathbf{x}) = \boldsymbol{\nabla} \varphi + (\mathbf{v}_{\text{in}} (\boldsymbol{\xi}) \boldsymbol{\nabla}) \boldsymbol{\xi} \tag{1.8}$$

where the quantities $\varphi, \boldsymbol{\xi}, \rho$ are functions of $t, \mathbf{x}$, satisfying the five equations

$$\partial_0 \xi_\alpha + \left(\boldsymbol{\nabla} \varphi + v_{\text{in}}^\beta (\boldsymbol{\xi}) \boldsymbol{\nabla} \xi_\beta\right) \boldsymbol{\nabla} \xi_\alpha = -\frac{\omega (t, \boldsymbol{\xi})}{\rho_{\text{in}} (\boldsymbol{\xi})} \varepsilon_{\alpha\beta\gamma} \Omega_{\text{in}}^{\beta\gamma} (\boldsymbol{\xi}), \qquad \alpha = 1,2,3 \tag{1.9}$$

$$\partial_0 \varphi + \frac{(\boldsymbol{\nabla} \varphi)^2}{2} - \frac{(v_{\text{in}}^\alpha (\boldsymbol{\xi}) \boldsymbol{\nabla} \xi_\alpha)^2}{2} + \frac{\partial (\rho E (\rho))}{\partial \rho} = \frac{\omega (t, \boldsymbol{\xi})}{\rho_{\text{in}} (\boldsymbol{\xi})} \mathbf{v}_{\text{in}} (\boldsymbol{\xi}) \boldsymbol{\Omega}_{\text{in}} (\boldsymbol{\xi}) \tag{1.10}$$

$$\partial_0 \rho + \boldsymbol{\nabla} (\rho (\boldsymbol{\nabla} \varphi + v_{\text{in}}^\alpha (\boldsymbol{\xi}) \boldsymbol{\nabla} \xi_\alpha)) = 0 \tag{1.11}$$

The equations (1.9), (1.10), (1.11) should be solved at the initial conditions

$$\boldsymbol{\xi} (0, \mathbf{x}) = \boldsymbol{\xi}_{\text{in}} (\mathbf{x}) = \mathbf{x}, \qquad \varphi (0, \mathbf{x}) = \varphi_{\text{in}} (\mathbf{x}) = 0, \qquad \rho (0, \mathbf{x}) = \rho_{\text{in}} (\mathbf{x}) \tag{1.12}$$

The quantity $E = E (\rho)$ is the fluid internal energy per unit mass. The quantities $\boldsymbol{\Omega}_{\text{in}} (\boldsymbol{\xi})$ are defined by the relations

$$\Omega_{\text{in}}^{\beta\gamma} (\boldsymbol{\xi}) = \left(\frac{\partial v_{\text{in}}^\beta (\boldsymbol{\xi})}{\partial \xi_\gamma} - \frac{\partial v_{\text{in}}^\gamma (\boldsymbol{\xi})}{\partial \xi_\beta}\right), \qquad \beta, \gamma = 1,2,3 \tag{1.13}$$

$$\boldsymbol{\Omega}_{\text{in}} (\boldsymbol{\xi}) = \left\{\Omega_{\text{in}}^{23} (\boldsymbol{\xi}), \Omega_{\text{in}}^{31} (\boldsymbol{\xi}), \Omega_{\text{in}}^{12} (\boldsymbol{\xi})\right\} \tag{1.14}$$

The quantity $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Chivita pseudotensor. The quantity $\omega = \omega (t, \boldsymbol{\xi})$ is an indefinite quantity, which is not determined from initial conditions. The fact, that the expressions (1.8) together with (1.9) - (1.14) are solutions of the system (1.1), (1.2), can be tested by means of the direct substitution and subsequent identical transformations. The indefinite quantity ω disappears as a result of these transformations. Appearance of indefinite quantity in equations (1.9), (1.10) and, hence, in (1.8) means that the the solution of the Cauchy problem for the Euler system (1.1), (1.2) is not unique. How is it possible? How can the Lin constraints (1.4) influence on the solution of the closed system of dynamic equations, which does not refer to variables $\boldsymbol{\xi}$ and Lin constraints?

We try to resolve this problem and to understand, what is a real reason of nonuniqueness of solution of the Cauchy problem. We start from the point, that the ideal barotropic fluid is a dynamical system, whose dynamic equations can be obtained from the variational principle.

2 Generalized stream function

Let us note that the quantities $\boldsymbol{\xi}$ may be considered to be the generalized stream function (GSF), because $\boldsymbol{\xi}$ have two main properties of the stream function.

1. GSF $\boldsymbol{\xi}$ labels world lines of fluid.

2. Some combinations of the first derivatives of any $\boldsymbol{\xi}$ satisfy the continuity equation identically.

$$\partial_k j^k \equiv 0, \qquad j^k = \frac{\partial J_{\xi/x}}{\partial \xi_{0,k}}, \qquad \partial_k \equiv \frac{\partial}{\partial x^k}, \qquad k = 0,1,2,3 \tag{2.1}$$

where $j^k = \{j^0, j^1, j^2, j^3\} = \{\rho, \rho\mathbf{v}\}$ is the 4-vector of flux. Here and in what follows, a summation over two repeated indices is produced (0-3) for Latin indices and (1-3) for Greek ones. The Jacobian determinant $J = J_{\xi/x}$

$$J_{\xi/x} = J\left(\xi_{l,k}\right) = \frac{\partial\left(\xi_0, \xi_1, \xi_2, \xi_3\right)}{\partial\left(x^0, x^1, x^2, x^3\right)} = \det\left|\left|\xi_{l,k}\right|\right|, \qquad \xi_{l,k} \equiv \frac{\partial \xi_l}{\partial x^k} \qquad l, k = 0,1,2,3 \tag{2.2}$$

is considered to be a four-linear function of $\xi_{l,k}$. The quantity ξ_0 is the temporal Lagrangian coordinate, which appears to be fictitious in expressions for the flux 4-vector j^k

$$\begin{aligned}
\rho &= j^0 = \frac{\partial\left(\xi_1, \xi_2, \xi_3\right)}{\partial\left(x^1, x^2, x^3\right)}, \qquad \rho v^1 = j^1 = -\frac{\partial\left(\xi_1, \xi_2, \xi_3\right)}{\partial\left(t, x^2, x^3\right)}, \\
\rho v^2 &= j^2 = \frac{\partial\left(\xi_1, \xi_2, \xi_3\right)}{\partial\left(t, x^1, x^3\right)}, \qquad \rho v^3 = j^3 = -\frac{\partial\left(\xi_1, \xi_2, \xi_3\right)}{\partial\left(t, x^1, x^2\right)}
\end{aligned} \tag{2.3}$$

A use of Jacobians in the description of the ideal fluid goes up to Clebsch [3, 4], who used Jacobians in the expanded form. It was rather bulky. We use a more rational designations, when the 4-flux and other essential dynamic quantities are presented in the form of derivatives of the principal Jacobian J. Dealing with the generalized stream function, the following identities are useful

$$\frac{\partial J}{\partial \xi_{i,l}} \xi_{k,l} \equiv J\delta_k^i, \qquad \partial_k \frac{\partial J}{\partial \xi_{0,k}} \equiv 0, \qquad \partial_l \frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{i,l}} \equiv 0 \tag{2.4}$$

$$\frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{l,s}} \equiv J^{-1}\left(\frac{\partial J}{\partial \xi_{0,k}} \frac{\partial J}{\partial \xi_{l,s}} - \frac{\partial J}{\partial \xi_{0,s}} \frac{\partial J}{\partial \xi_{l,k}}\right) \tag{2.5}$$

See details of working with Jacobians and the generalized stream functions in [5].

Example. Application of the stream function for integration of equations, describing the 2D stationary flow of incompressible fluid.

Dynamic equations have the form

$$u_x + v_y = 0, \qquad \partial_y\left(uu_x + vu_y\right) = \partial_x\left(uv_x + vv_y\right) \tag{2.6}$$

where u and v are velocity components along x-axis and y-axis respectively.

Introducing the stream function ψ by means of relations

$$u = -\psi_y, \qquad v = \psi_x \tag{2.7}$$

we satisfy the first equation (2.6) identically and obtain for the second equation (2.6) the relations

$$\begin{aligned}
\psi_y \psi_{xyy} - \psi_x \psi_{yyy} &= -\psi_y \psi_{xxx} + \psi_x \psi_{xxy} \\
\psi_y \left(\psi_{xyy} + \psi_{xxx}\right) &= \psi_x \left(\psi_{xxy} + \psi_{yyy}\right)
\end{aligned}$$

which can be rewritten in the form

$$\frac{\partial (\omega, \psi)}{\partial (x, y)} = 0, \qquad \omega \equiv \psi_{xx} + \psi_{yy} \tag{2.8}$$

where ω is the vorticity of the fluid flow. The general solution of equation (2.8) has the form

$$\omega = \psi_{xx} + \psi_{yy} = \Omega(\psi) \tag{2.9}$$

where Ω is an arbitrary function of ψ.

For the irrotational flow the vorticity $\Omega(\psi) = 0$, and we obtain instead (2.9)

$$\psi_{xx} + \psi_{yy} = 0 \tag{2.10}$$

One obtains the unique solution of (2.10) inside of a closed region of 2D space provided, that the value $\psi|_\Sigma$ of the stream function ψ is given on the boundary Σ of this region. The differential structure of equations (2.9) and (2.10) is similar. One should expect, that giving the value $\psi|_\Sigma$ of the stream function ψ on the boundary Σ, one obtains the unique solution of the equation (2.10). But it is not so, because the indefinite function $\Omega(\psi)$ is not given, and it cannot be determined from the boundary condition, because the nature of the function $\Omega(\psi)$ is another, than the nature of the boundary conditions. First, if the flow contains closed stream lines, which do not cross the boundary, one cannot determine the values of Ω on these stream lines from the boundary conditions. But for determination of the unique solution the values of Ω on the closed stream lines must be given. Second, boundary conditions are given arbitrarily. The function Ω cannot be given arbitrarily. For those stream lines, which cross the boundary more than once, the values of Ω on the different segments of the boundary are to be agreed. Thus, the nonuniqueness of the solution, connected with the indefinite function Ω has another nature, than the nonuniqueness, connected with the insufficiency of the boundary conditions.

3 Derivation of hydrodynamic equations from the variational principle

We use the variational principle for the derivation of the hydrodynamic equations (1.1), (1.2), (1.4). The action functional has the form

$$\mathcal{A}[\xi, j, p] = \int\limits_{V_x} \left\{ \frac{\mathbf{j}^2}{2\rho} - \rho E(\rho) - p_k \left(j^k - \rho_0(\boldsymbol{\xi}) \frac{\partial J}{\partial \xi_{0,k}} \right) \right\} d^4x, \tag{3.1}$$

where p_k, $k=0,1,2,3$ are the Lagrange multipliers, introducing the designations for the 4-flux

$$j^k=\rho_0(\boldsymbol{\xi})\frac{\partial J}{\partial \xi_{0,k}},\qquad k=0,1,2,3 \tag{3.2}$$

Note, the expression for the 4-flux (3.2) satisfies the first equation (2.1) identically, because the expression (3.2) may be reduced to the form of the second relation (2.1) by means of a change of variables $\boldsymbol{\xi}$

$$\tilde{\xi}_0=\xi_0,\qquad \tilde{\xi}_1=\int\rho_0(\boldsymbol{\xi})\,d\xi_1,\qquad \tilde{\xi}_2=\xi_2,\qquad \tilde{\xi}_3=\xi_3$$

Then

$$\rho_0(\boldsymbol{\xi})\frac{\partial(\xi_0,\xi_1,\xi_2,\xi_3)}{\partial(x^0,x^1,x^2,x^3)}=\frac{\partial\left(\tilde{\xi}_0,\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3\right)}{\partial(x^0,x^1,x^2,x^3)},\qquad \tilde{\xi}_1=\int\rho_0(\boldsymbol{\xi})\,d\xi_1$$

Variation with respect to $p_k=\{p_0,\mathbf{p}\}$ gives relations (3.2). Another dynamic equations have the form

$$\delta\rho:\qquad p_0=-\frac{\mathbf{j}^2}{2\rho}-\frac{\partial}{\partial\rho}(\rho E(\rho))=-\frac{\mathbf{v}^2}{2}-\frac{\partial}{\partial\rho}(\rho E) \tag{3.3}$$

$$\delta\mathbf{j}:\qquad \mathbf{p}=\frac{\mathbf{j}}{\rho}=\mathbf{v} \tag{3.4}$$

$$\delta\xi_l:\qquad -\partial_s\left(\rho_0(\boldsymbol{\xi})\,p_k\frac{\partial^2 J}{\partial\xi_{0,k}\partial\xi_{l,s}}\right)+p_k\frac{\partial\rho_0}{\partial\xi_l}(\boldsymbol{\xi})\frac{\partial J}{\partial\xi_{0,k}}=0,\qquad l=0,1,2,3 \tag{3.5}$$

Using the third relation (2.4), we obtain

$$-\frac{\partial^2 J}{\partial\xi_{0,k}\partial\xi_{l,s}}\left(\frac{\partial\rho_0(\boldsymbol{\xi})}{\partial\xi_\alpha}\xi_{\alpha,s}+\rho_0(\boldsymbol{\xi})\,\partial_s p_k\right)+p_k\frac{\partial\rho_0}{\partial\xi_l}(\boldsymbol{\xi})\frac{\partial J}{\partial\xi_{0,k}}=0 \tag{3.6}$$

Now using (2.5), we obtain

$$-J^{-1}\left(\frac{\partial J}{\partial\xi_{0,k}}\frac{\partial J}{\partial\xi_{l,s}}-\frac{\partial J}{\partial\xi_{0,s}}\frac{\partial J}{\partial\xi_{l,k}}\right)\left(\frac{\partial\rho_0(\boldsymbol{\xi})}{\partial\xi_\alpha}\xi_{\alpha,s}+\rho_0(\boldsymbol{\xi})\,\partial_s p_k\right)+p_k\frac{\partial\rho_0}{\partial\xi_l}(\boldsymbol{\xi})\frac{\partial J}{\partial\xi_{0,k}}=0 \tag{3.7}$$

Using the first relation (2.4), we obtain

$$J^{-1}\left(\frac{\partial J}{\partial\xi_{0,k}}\frac{\partial J}{\partial\xi_{l,s}}-\frac{\partial J}{\partial\xi_{0,s}}\frac{\partial J}{\partial\xi_{l,k}}\right)\rho_0(\boldsymbol{\xi})\,\partial_s p_k=0,\qquad l=0,1,2,3 \tag{3.8}$$

There are two ways of dealing with this equation:

1. Elimination of GSF $\boldsymbol{\xi}$, which leads to the Euler equations.
2. Integration, which leads to appearance of arbitrary functions.

The first way: elimination of GSF

Convoluting (3.8) with $\xi_{l,i}$ and using dynamic equations (3.2), we obtain

$$j^k\partial_i p_k-j^k\partial_k p_i=0,\qquad i=0,1,2,3 \tag{3.9}$$

Substituting p_k and j^k from relations (3.3) and (3.4), we obtain the Euler dynamic equations (1.1)

$$\partial_0 v^\alpha + (\mathbf{v}\boldsymbol{\nabla}) v^\alpha = -\partial_\alpha \frac{\partial}{\partial\rho}(\rho E) = -\frac{1}{\rho}\partial_\alpha p, \qquad \alpha = 1,2,3, \qquad p = \rho^2 \frac{\partial E}{\partial \rho} \tag{3.10}$$

The continuity equation (1.2) is a corollary of equations (3.2) and identity (2.1).

Finally the Lin constraints (3.4) are corollaries of the first identity (2.4) and dynamic equations (3.2), (3.3), (3.4).

The second way: integration of the equation for p_k

Let us consider the equations (3.8) as linear differential equations for p_k. The general solution of (3.8) has the form

$$p_k = (\partial_k \varphi + g^\alpha(\boldsymbol{\xi})\,\partial_k \xi_\alpha), \qquad k = 0,1,2,3 \tag{3.11}$$

where $g^\alpha(\boldsymbol{\xi})$, $\alpha = 1,2,3$ are arbitrary functions of $\boldsymbol{\xi}$, $\varphi = g^0(\xi_0)$ is a new variable instead of fictitious variable ξ_0. Substituting expressions

$$\partial_s p_k = (\partial_s \partial_k \varphi + g^\alpha(\boldsymbol{\xi})\,\partial_s \partial_k \xi_\alpha) + \frac{\partial g^\alpha(\boldsymbol{\xi})}{\partial \xi_\beta}\partial_k \xi_\alpha \partial_s \xi_\beta \tag{3.12}$$

in (3.8) and using the first identity (2.4), we see, that the relations (3.12) satisfy the equations (3.8) identically.

We may substitute (3.11) in the action (3.1), or introduce (3.11) by means of the Lagrange multipliers. (the result is the same). We obtain the new action functional

$$\mathcal{A}[\xi, j] = \int\limits_{V_x} \left\{ \frac{\mathbf{j}^2}{2\rho} - \rho E(\rho) - j^k (\partial_k \varphi + g^\alpha(\boldsymbol{\xi})\,\partial_k \xi_\alpha) \right\} d^4 x, \tag{3.13}$$

where

$$j^0 = \rho, \qquad \mathbf{j} = \rho \mathbf{v} = \{j^1, j^2, j^3\} \tag{3.14}$$

The term

$$p_k \frac{\partial J}{\partial \xi_{0,k}} = (\partial_k \varphi + g^\alpha(\boldsymbol{\xi})\,\partial_k \xi_\alpha) \frac{\partial J}{\partial \xi_{0,k}} = \frac{\partial(\varphi, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)} \tag{3.15}$$

is omitted, because it does not contribute to dynamic equations.

Variation of (3.13) with respect to φ, ρ and j^μ gives respectively

$$\delta\varphi: \qquad \partial_k j^k = 0 \tag{3.16}$$

$$\delta\rho: \qquad \partial_0 \varphi + g^\beta(\boldsymbol{\xi})\,\partial_0 \xi_\beta + \frac{\mathbf{j}^2}{2\rho^2} + \frac{\partial}{\partial\rho}(\rho E(\rho)) = 0 \tag{3.17}$$

$$\delta j^\mu: \qquad v^\mu = \frac{j^\mu}{\rho} = \partial_\mu \varphi + g^\alpha(\boldsymbol{\xi})\,\partial_\mu \xi_\alpha \tag{3.18}$$

Variation of (3.13) with respect to ξ_α gives

$$\delta\xi_\alpha: \qquad \rho\Omega^{a\mu}(\boldsymbol{\xi})\,(\partial_0 \xi_\alpha + (\mathbf{v}\boldsymbol{\nabla})\,\xi_\alpha) = 0, \tag{3.19}$$

$$\Omega^{a\mu}(\boldsymbol{\xi}) = \left(\frac{\partial g^{\alpha}(\boldsymbol{\xi})}{\partial \xi_{\mu}} - \frac{\partial g^{\mu}(\boldsymbol{\xi})}{\partial \xi_{\alpha}}\right) \tag{3.20}$$

If $\det\left|\left|\Omega^{\alpha\beta}\right|\right| \neq 0$, then the Lin conditions

$$(\partial_0 \xi_{\alpha} + (\mathbf{v}\boldsymbol{\nabla})\,\xi_{\alpha}) = 0 \tag{3.21}$$

follows from (3.19)

However, the matrix $\Omega^{\alpha\beta}$ is antisymmetric and

$$\det\left|\left|\Omega^{\alpha\beta}\right|\right| = \begin{vmatrix} 0 & \Omega^{12} & \Omega^{13} \\ -\Omega^{12} & 0 & \Omega^{23} \\ -\Omega^{13} & -\Omega^{23} & 0 \end{vmatrix} \equiv 0 \tag{3.22}$$

Then it follows from (3.19)

$$\partial_0 \xi_{\alpha} + \mathbf{v}\boldsymbol{\nabla}\xi_{\alpha} = -\frac{\omega}{\rho_0(\boldsymbol{\xi})}\varepsilon_{\alpha\beta\gamma}\Omega^{\beta\gamma}(\boldsymbol{\xi}) \qquad \alpha = 1,2,3 \tag{3.23}$$

where $\omega = \omega(t,\boldsymbol{\xi})$ is an arbitrary quantity, and $\rho_0(\boldsymbol{\xi})$ is the weight function from (3.2).

Note, that eliminating the variables φ and $\boldsymbol{\xi}$ from dynamic equations (3.17) - (3.19), we obtain the Euler dynamic equations (1.1).

The vorticity $\boldsymbol{\omega}_0 \equiv \nabla \times \mathbf{v}$ and $\mathbf{v} \times \boldsymbol{\omega}_0$ are obtained from (3.18) in the form

$$\boldsymbol{\omega}_0 = \boldsymbol{\nabla} \times \mathbf{v} = \frac{1}{2}\Omega^{\alpha\beta}\boldsymbol{\nabla}\xi_{\beta} \times \boldsymbol{\nabla}\xi_{\alpha} \tag{3.24}$$

$$\mathbf{v} \times \boldsymbol{\omega}_0 = \Omega^{\alpha\beta}\boldsymbol{\nabla}\xi_{\beta}(\mathbf{v}\boldsymbol{\nabla})\xi_{\alpha} \tag{3.25}$$

Let us form a difference between the time derivative of (3.18) and the gradient of (3.17). Eliminating $\Omega^{\alpha\beta}\partial_0\boldsymbol{\xi}_{\alpha}$ by means of equations (3.20), one obtains

$$\partial_0\mathbf{v} + \boldsymbol{\nabla}\frac{\mathbf{v}^2}{2} + \frac{\partial^2(\rho E)}{\partial \rho^2}\boldsymbol{\nabla}\rho - \Omega^{\alpha\beta}\boldsymbol{\nabla}\xi_{\beta}(\mathbf{v}\boldsymbol{\nabla})\xi_{\alpha} = 0 \tag{3.26}$$

Using (3.25) and (3.24), the expression (3.26) reduces to

$$\partial_0\mathbf{v} + \boldsymbol{\nabla}\frac{\mathbf{v}^2}{2} + \frac{1}{\rho}\boldsymbol{\nabla}(\rho^2\frac{\partial E}{\partial \rho}) - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = 0 \tag{3.27}$$

In virtue of the identity

$$\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) \equiv \boldsymbol{\nabla}\frac{\mathbf{v}^2}{2} - (\mathbf{v}\boldsymbol{\nabla})\mathbf{v} \tag{3.28}$$

the last equation is equivalent to (1.1).

Note, that the Euler equations (1.1) are obtained at any form of the arbitrary function $\omega(t,\boldsymbol{\xi})$ in the equations (3.23), because the equations (3.23) are used in the form (3.19), where the form of $\omega(t,\boldsymbol{\xi})$ is unessential.

If $\omega(t,\boldsymbol{\xi}) \neq 0$, the dynamic equations (3.23) describe a violation of the Lin constraints (1.4). The transformation (3.23) of the Lin constraints means a change of evolution of the fluid labeling. Note that relabeling (1.6) does not violate the Lin constraints. Let $\mathbf{v}$ be a solution of Lin constraints (1.4), considered as equations for determination of $\mathbf{v}$. Then

$$v^{\mu} = \left(\frac{\partial J_{\xi/x}}{\partial \xi_{0,0}}\right)^{-1} \frac{\partial J_{\xi/x}}{\partial \xi_{0,\mu}}, \qquad \mu = 1,2,3, \qquad J_{\xi/x} = \frac{\partial(\xi_0,\xi_1,\xi_2,\xi_3)}{\partial(x^0,x^1,x^2,x^3)} \tag{3.29}$$

After transformation

$$\xi_0 \to \tilde{\xi}_0 = \xi_0, \qquad \boldsymbol{\xi} \to \tilde{\boldsymbol{\xi}} = \tilde{\boldsymbol{\xi}}(\boldsymbol{\xi}), \qquad J_{\tilde{\boldsymbol{\xi}}/\boldsymbol{\xi}} = \frac{\partial\left(\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3\right)}{\partial(\xi_1,\xi_2,\xi_3)} \neq 0 \tag{3.30}$$

the velocity $\mathbf{v} = \{v^1, v^2, v^3\}$ is transformed to $\tilde{\mathbf{v}} = \{\tilde{v}^1, \tilde{v}^2, \tilde{v}^3\}$

$$v^{\mu} \to \tilde{v}^{\mu} = \left(\frac{\partial J_{\tilde{\xi}/x}}{\partial \tilde{\xi}_{0,0}}\right)^{-1} \frac{\partial J_{\tilde{\xi}/x}}{\partial \tilde{\xi}_{0,\mu}}, \qquad \mu = 1,2,3, \qquad J_{\tilde{\xi}/x} = \frac{\partial\left(\tilde{\xi}_0,\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3\right)}{\partial(x^0,x^1,x^2,x^3)} \tag{3.31}$$

As far as

$$\frac{\partial J_{\tilde{\xi}/x}}{\partial \tilde{\xi}_{0,k}} = J_{\tilde{\boldsymbol{\xi}}/\boldsymbol{\xi}} \frac{\partial J_{\xi/x}}{\partial \xi_{0,k}}, \qquad k = 0,1,2,3, \qquad J_{\tilde{\boldsymbol{\xi}}/\boldsymbol{\xi}} = \frac{\partial\left(\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3\right)}{\partial(\xi_1,\xi_2,\xi_3)} \tag{3.32}$$

the velocity $\mathbf{v}$ is invariant with respect to the relabeling (3.30)

$$v^{\mu} = \tilde{v}^{\mu} = \left(\frac{\partial J_{\tilde{\xi}/x}}{\partial \tilde{\xi}_{0,0}}\right)^{-1} \frac{\partial J_{\tilde{\xi}/x}}{\partial \tilde{\xi}_{0,\mu}}, \qquad \mu = 1,2,3 \tag{3.33}$$

If the velocity $\mathbf{v}$ is given as a function of $t, \mathbf{x}$, one can determine the labeling $\tilde{\boldsymbol{\xi}}$ evolution, solving the Lin constraints (1.4) with respect to $\tilde{\boldsymbol{\xi}}$ with initial conditions $\tilde{\boldsymbol{\xi}}(0,\mathbf{x}) = \mathbf{x}$

$$\partial_0 \tilde{\xi}_{\alpha} + (\mathbf{v}\boldsymbol{\nabla})\,\tilde{\xi}_{\alpha} = 0, \qquad \tilde{\xi}_{\alpha}(0,\mathbf{x}) = x^{\alpha}, \qquad \alpha = 1,2,3 \tag{3.34}$$

If the velocity $\mathbf{v}$ is defined by relations (1.8) - (1.13), it satisfies the Euler equations and associates with the generalized stream function $\boldsymbol{\xi}(t,\mathbf{x})$, whose evolution is described by the equations (3.23)

$$\partial_0 \xi_{\alpha} + \mathbf{v}\boldsymbol{\nabla}\xi_{\alpha} = -\frac{\omega}{\rho_0(\boldsymbol{\xi})}\varepsilon_{\alpha\beta\gamma}\Omega^{\beta\gamma}(\boldsymbol{\xi}), \qquad \xi_{\alpha}(0,\mathbf{x}) = x^{\alpha}, \qquad \alpha = 1,2,3 \tag{3.35}$$

In general, the evolution of the quantities $\tilde{\boldsymbol{\xi}}$ and $\boldsymbol{\xi}$ is different. Let

$$\boldsymbol{\eta} = \boldsymbol{\xi} - \tilde{\boldsymbol{\xi}} \tag{3.36}$$

It follows from (3.34) and (3.35) that mismatch $\boldsymbol{\eta}$ between $\tilde{\boldsymbol{\xi}}$ and $\boldsymbol{\xi}$ is determined by the relation

$$\partial_0\eta_\alpha+\mathbf{v}\left(t,\mathbf{x}\right)\boldsymbol{\nabla}\eta_\alpha+\frac{\omega\left(t,\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)}{\rho_0\left(\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)}\varepsilon_{\alpha\beta\gamma}\Omega^{\beta\gamma}\left(\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)=0,\qquad\eta_\alpha\left(0,\mathbf{x}\right)=0,\qquad\alpha=1,2,3 \tag{3.37}$$

The system of ordinary differential equations, associated with the equation (3.37), has the form

$$\frac{dt}{1}=\frac{dx^1}{v^1\left(t,\mathbf{x}\right)}=\frac{dx^2}{v^2\left(t,\mathbf{x}\right)}=\frac{dx^3}{v^3\left(t,\mathbf{x}\right)}=\frac{\rho_0\left(\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)d\eta_\alpha}{\omega\left(t,\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)\varepsilon_{\alpha\beta\gamma}\Omega^{\beta\gamma}\left(\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)},\qquad\alpha=1,2,3 \tag{3.38}$$

Solution of the system of ordinary equations at the initial conditions $\boldsymbol{\eta}\left(0,\mathbf{x}\right)=0$ has the form

$$\eta_\mu\left(t,\mathbf{x}\right)=\int\limits_0^t\frac{\omega\left(t,\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)\varepsilon_{\mu\beta\gamma}}{\rho_0\left(\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)}\Omega^{\beta\gamma}\left(\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)dt,\qquad\mu=1,2,3 \tag{3.39}$$

Then

$$\tilde{\xi}_\mu\left(t,\mathbf{x}\right)=\xi_\mu\left(t,\mathbf{x}\right)-\int\limits_0^t\frac{\omega\left(t,\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)\varepsilon_{\mu\beta\gamma}}{\rho_0\left(\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)}\Omega^{\beta\gamma}\left(\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)dt,\qquad\mu=1,2,3 \tag{3.40}$$

Thus, the quantities $\boldsymbol{\xi}$ in relations (1.8) - (1.13) is not a real generalized stream function, because the variables ξ_1,ξ_2,ξ_3 do not label, in general, world lines of the fluid particles. The variables $\boldsymbol{\xi}$ carry out, in general, a pseudo-labeling. The real labeling of the fluid world lines is carried out by variables $\tilde{\boldsymbol{\xi}}$, satisfying the Lin constraints (3.34). Pseudo-labeling $\boldsymbol{\xi}$ coincides with the real labeling $\tilde{\boldsymbol{\xi}}$, if $\omega\left(t,\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)\equiv0$. In this case we have the unique solution of the Cauchy problem for the vortical 3D flow of the barotropic fluid.

In the general case, when $\tilde{\boldsymbol{\xi}}\neq\boldsymbol{\xi}$ and pseudo-labeling $\boldsymbol{\xi}$ "floats", we have the unique solution of the Cauchy problem for the Euler equations, provided the pseudo-labeling $\boldsymbol{\xi}$ is fixed, i.e. the form of function $\omega\left(t,\boldsymbol{\xi}\left(t,\mathbf{x}\right)\right)$ is fixed.

Existence of pseudo-labeling $\boldsymbol{\xi}$ puts a very important question, whether a solution of the hydrodynamic equations (1.1), (1.2) with $\tilde{\boldsymbol{\xi}}\neq\boldsymbol{\xi}$ describes a real flow of the barotropic fluid, or the solution of the Cauchy problem describes a real flow only in the case, when $\tilde{\boldsymbol{\xi}}=\boldsymbol{\xi}$. It is a difficult problem, which needs a further investigation.

We believe, that any solution of the Euler equations describes a real flow of the barotropic fluid, because the labeling procedure is rather conditional, when the "fluid particle" contains many molecules, which can pass from one fluid particle to another. One. one can see this intermixing effect in the figure, where dashed lines show real trajectories of the gas particles, whereas solid lines show lines of the constant labeling $\boldsymbol{\xi}$. In the figure the intermixing effect is rather rough. In the considered case of the Cauchy problem this effect is infinitesimal. It increases with increase of time.

Remark. We do not think, that the pseudo-labeling is the only reason, why the solution of the Cauchy problem is not unique, because the example of the 2D stationary flow of incompressible fluid shows some designs of insufficiency of boundary conditions.

4 Two-dimensional vortical flow of ideal barotropic fluid

It seems, that in the two-dimensional flow instead of determinant (3.22) we have the determinant

$$\left\| \begin{array}{ll} 0 & \Omega^{12} \\ -\Omega^{12} & 0 \end{array} \right\| = \left(\Omega^{12}\right)^2 \tag{4.1}$$

which does not vanish, in general. Then the problem of pseudo-labeling is removed and the solution of the Cauchy problem becomes to be unique.

In reality, we may control the solution only via initial conditions. We may give the two-dimensional initial conditions, i.e.

$$\partial_3 \mathbf{v}_{\text{in}}(\mathbf{x}) = 0, \qquad \partial_3 \rho_{\text{in}}(\mathbf{x}) = 0, \qquad v_{\text{in}}^3(\mathbf{x}) = 0 \tag{4.2}$$

In this case

$$\Omega_{\text{in}}^{12}(\xi_1, \xi_2) = \frac{\partial v_{\text{in}}^1(\xi_1, \xi_2)}{\partial \xi_2} - \frac{\partial v_{\text{in}}^2(\xi_1, \xi_2)}{\partial \xi_1}, \qquad \Omega_{\text{in}}^{23}(\boldsymbol{\xi}) = 0, \qquad \Omega_{\text{in}}^{31}(\boldsymbol{\xi}) = 0 \tag{4.3}$$

The determinant

$$\det \left\|\Omega^{\alpha\beta}\right\| = \left| \begin{array}{ccc} 0 & \Omega^{12} & 0 \\ -\Omega^{12} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right| \equiv 0 \tag{4.4}$$

and the relations (3.23) take the form

$$\partial_0 \xi_1 + \mathbf{v}(t, \mathbf{x}) \boldsymbol{\nabla} \xi_1 = 0, \qquad \partial_0 \xi_2 + \mathbf{v}(t, \mathbf{x}) \boldsymbol{\nabla} \xi_2 = 0 \tag{4.5}$$

$$\partial_0 \xi_3 + \mathbf{v}(t, \mathbf{x}) \boldsymbol{\nabla} \xi_3 = -\frac{\omega(t, \boldsymbol{\xi})}{\rho_0(\xi_1, \xi_2)} \Omega^{1,2}(\xi_1, \xi_2) \tag{4.6}$$

One cannot control indefinite quantity $\omega(t, \boldsymbol{\xi})$, which may depend on x^3. The equation (4.6) generates the problem of pseudo-labeling and the 3D vortical flow. The flow with the two-dimensional initial conditions turns into three-dimensional vortical flow.

5 Conclusions

Solution of the Cauchy problem for the vortical flow of ideal barotropic fluid is not unique, if we solve only the Euler system of hydrodynamic equations without consideration of the Lin constraints.

Solution of the Cauchy problem for the vortical flow of ideal barotropic fluid is unique, if we solve the Euler system of dynamic equations together with the Lin constraints.

The question is open, whether the Euler system of hydrodynamic equations describes always a real flow of ideal barotropic fluid, .

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CAPTIONS

Figure 1. Dashed lines show real trajectories of particles. The solid lines show trajectories of the mean particle motion.